A THEOREM REVISITED – WAVE FRONT CURVATURE IN MEDIA WITH LAYER-INDUCED ANISOTROPY

BY

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1. INTRODUCTION

The invitation to write for this volume caused me to re-read the article I published together with Th. Krey nearly twenty-five years ago (Krey and Helbig 1956). The article essentially consisted of:

(i) an inequality concerning curvature, that states that, for anisotropy caused by layering with isotropic constituents, the curvature of the quasi-longitudinal wave front in the vicinity of the axis of symmetry is at least equal to the curvature of the ellipsoid with the same axes;

(ii) the principal theorem, that states that – provided the ratio v_s/v_l is the same for all layers – in the vicinity of the axis of symmetry the wave front of compressional waves deviates from a co-centric sphere only by terms of fourth and higher order; and

(iii) a computation of corrections for greater dips, which dealt with the position and orientation of reflecting elements (this was my contribution to the paper).

In the present paper I call a medium that satisfies (ii) a K-medium.

Today one could say that the inequality is more significant than the theorem, because the ratio of transverse to longitudinal velocities is no longer *assumed,* but *measured* in wells, and it does not appear to be constant to the expected degree. Moreover, there is a certain reluctance to abandon elliptical wave fronts for simple calculations. This might be due to some degree to the phrase 'at least equal' in the inequality. It seems, therefore, important to strengthen the inequality. The expressions in the last section, on the other hand, dealt with the positioning of reflection elements only. A few years after the article was published, the determination of stacking velocities from reflection data became the standard of the industry. The calculation of arrival times is implicit in the expressions but somewhat inconvenient to carry out. Therefore, a re-formulation of these expressions – preferably in terms of a parameter more practical than Poisson's ratio – seemes to be indicated.

A further reason to look at the matter again is because horizontally polarized shear waves have come into use in recent years. The full potential of reflection seismics using shear waves lies in the combination with reflections of longitudinal waves. To this end the velocities of both wave types in the vertical direction are necessary. For the longitudinal waves, ordinary velocity determination – for instance from the slope of the regression line through the t^2-versus x^2 data-works, by virtue of the theorem, even if layering causes the material to be anisotropic, provided the velocity ratio (or Poisson's ratio) does not fluctuate too much. However, horizontally polarized shear waves have an elliptical wave front, and for such a wave front the slope of the regression line corresponds to the *horizontal* velocity (e.g., see Levin 1978 and Helbig 1979).

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The radius of curvature of the quasi-longitudinal wave front in the vicinity of the axis of symmetry of *any* transversely isotropic medium is

$$r_P = r_K \cdot \frac{c_{44} + \dfrac{(c_{13}+c_{44})^2}{c_{33}-c_{44}}}{c_{33}}, \text{ where } \delta \text{ is the density} \tag{1}$$

and $r_K = t \cdot \sqrt{(c_{33}/\delta)}$ is the radius for the K-Medium, i.e., for constant $\theta = (v_s/v_l)^2$. According to the theorem the radius of curvature of the quasi-longitudinal wave front for a K-medium is equal to that of the concentric sphere. For the ellipsoid with the axes $t \cdot \sqrt{(c_{33}/\delta)}$ and $t \cdot \sqrt{(c_{11}/\delta)}$ we have

$$r_E/r_K = c_{11}/c_{33} \tag{2}$$

and thus

$$\frac{r_E}{r_K} - \frac{r_P}{r_K} = \left[(c_{11}-c_{44}) - \frac{(c_{13}+c_{44})^2}{c_{33}-c_{44}}\right]\Big/ c_{33}. \tag{3}$$

The curvature inequality is shown to hold if it can be shown that the term in brackets is positive semi-definite, or if the "fundamental inequality" (Berryman 1979)

$$(c_{11}-c_{44}) \cdot (c_{33}-c_{44}) \geqq (c_{13}+c_{44})^2 \tag{4}$$

holds. In 1956 this inequality was known to hold for stratified media with two distinct constituents (Postma 1955). However, in 1962 it was shown that there are stratified media that cannot be modeled with just two constituents (Backus 1962), thus some doubt was cast on the general validity of the fundamental inequality. Meanwhile, two independent proofs have been published (Berryman 1979, 1980, Helbig 1979) showing that (4) is valid for any number of stable isotropic layers. Stability is guaranteed if $0 \leqq \theta < 3/4$).
Berryman's proof contains the key to a generalization and strengthening of the fundamental inequality.

2.1 BOUNDS FOR $(c_{11}-c_{44}) \cdot (c_{33}-c_{44}) - (c_{13}+c_{44})^2 = X^2 \cdot 4 \cdot c_{33} \cdot c_{44}$

To obtain bounds for any anisotropy parameter, one starts with the bounds on the five elastic parameters describing the transversely isotropic medium. These bounds have to be traced back to the bounds on the elastic parameters of the constituents. While, in principle, any set of five (independent) parameters for the transversely isotropic medium and any two (independent) parameters for the isotropic constituents would do, the simplicity of the proof depends crucially on this choise. "Conversion rules" serve to convert the chosen set of parameters to the common set.

Since we are interested in dimensionless ratios only, we can use arbitrarily normalized parameters, thus reducing the number for the transversely isotropic medium to four. The set we use is based on those introduced by Backus (1963):

$$\tau = \left\langle \frac{1}{2}\left(1-\frac{c_{13}}{c_{33}}\right)\right\rangle, \qquad \lambda = \frac{1}{\left\langle \frac{1}{c_{44}} \right\rangle \langle c_{66}\rangle}, \tag{5}$$

$$\rho = \left\langle \frac{1}{c_{33}} \right\rangle \Big/ \left\langle \frac{1}{c_{44}} \right\rangle, \text{ and } \sigma = \left\langle \frac{1}{4}\left(\frac{c_{13}{}^2}{c_{33}} - c_{11}\right) + c_{66} \right\rangle \Big/ \langle c_{66}\rangle.$$

Equations (5) do double duty as the "mixing rules" and as conversion rules. If used as mixing rules – i.e., for the calculation of the parameters λ, τ, ρ, σ of the compound medium from those of the constituents – the angled brackets have to be read as

$$\langle a \rangle = \sum p_i a_i \quad \text{with} \quad \sum p_i = 1,$$

in other words, as the weighted average of the argument. If used as conversion rules, the angled brackets are read as ordinary parentheses, e.g., $\lambda = c_{44}/c_{66}$. As mixing rules, the equations (1) could be applied even to anisotropic constituents. If the constituents are isotropic, we have

$$(1-c_{13}/c_{33})/2 = \theta, \qquad c_{44} = c_{66} = \mu,$$

$$1/c_{33} = \theta/\mu, \qquad \text{and} \qquad (c_{13}^2/c_{33} - c_{11})/4 + c_{66} = \theta \cdot \mu. \tag{6}$$

From (5) and (6) we have

$$\sum p_i \theta_i = \tau, \qquad \frac{1}{\sum p_i \mu_i \sum p_i/\mu_i} = \lambda \tag{7}$$

$$\frac{\sum p_i \frac{\theta_i}{\mu_i}}{\sum \frac{p_i}{\mu_i}} = \rho, \quad \text{and}$$

$$\frac{\sum p_i \theta_i \mu_i}{\sum p_i \mu_i} = \sigma.$$

It is obvious that ρ, σ, and τ are weighted averages of the θ_i and thus are constrained to the open interval (θ_l, θ_h), where θ_l and θ_h are the lowest and highest value for the constituents, respectively. The trivial case of a single constituent is excluded, and the interval (θ_l, θ_l) is defined as open. Thus we have

$$\begin{aligned} &\theta_l \leqq \theta \leqq \theta_h, \\ &\theta_l < \rho < \theta_h, \\ &\theta_l < \sigma < \theta_h, \quad \text{and} \\ &\theta_l < \tau < \theta_h. \end{aligned} \tag{8}$$

Further constraints are obtained with the help of the Cauchy-Schwartz inequality

$$(\Sigma p_i)^2 \leqq (\Sigma p_i \mu_i) \left(\Sigma \frac{p_i}{\mu_i}\right) \tag{9}$$

from which we have immediately

$$\lambda \leqq 1.$$

The equality sign prevails for $\mu_i \equiv \mu$, in which case we have from (7) also $\rho = \sigma = \tau$, i.e., the compound medium is isotropic. We exclude this case also and use henceforth $\lambda < 1$.
The Cauchy-Schwartz inequality can be re-written as (see Berryman 1980)

$$(\Sigma p_i (\beta - \theta_i))^2 < \frac{\Sigma p_i \mu_i (\beta - \theta_i)}{\Sigma p_i \mu_i} \; \frac{\Sigma \frac{p_i}{\mu_i} (\beta - \theta_i)}{\Sigma p_p / \mu_i} \; \Sigma p_i \mu_i \; \Sigma p_i / \mu_i. \tag{10}$$

Inequality (10) holds for any constant β that does not cause a change of sign in any argument (except a change of sign in *all* arguments), i.e., for

$$\beta \leqq \theta_l \text{ or } \theta_h \leqq \beta. \tag{11}$$

Substitution of (11) into (10) yields

$$\frac{1}{\lambda} (\theta_{l,h} - \rho)(\theta_{l,h} - \sigma) > (\theta_{l,h} - \tau)^2, \text{ where } \theta_{l,h} \text{ means "}\theta_l \text{ or } \theta_h\text{".} \tag{12}$$

The two inequalities (12) are 'strong' inequalities since we have excluded the case of constant μ. On the other hand, the difference between the two sides of the inequalities can be made arbitrarily small for any λ by choosing suitable (positive) $\mu > 0$ and suitable θ_i from the open interval (θ_l, θ_h) (see Backus 1963).
If we express (4) with the help of (5) in terms of λ, ρ, and τ, we obtain

$$(c_{11} - c_{44})(c_{33} - c_{44}) - (c_{13} + c_{44})^2 = 4 c_{44} c_{33} \left[\frac{1}{\lambda}(1-\rho)(1-\sigma) - (1-\tau)^2\right] > 0. \tag{13}$$

The inequality holds because the term in the square brackets is positive definite. This is easily seen from (12) if one replaces θ_h by $1 (> \theta_h)$. The inequality in (13) is also a strong inequality, but it differs essentially from the inequalities (12): for a λ differing distinctly from 1 the term in the square bracket cannot be made arbitrarily small – since $\theta_h < 3/4 < 1$ – in other words, there must be a lower bound different from 0. The equation

$$X^2 = [c_{11} - c_{44}) \cdot (c_{33} - c_{44}) - (c_{13} + c_{44})^2] / (4 c_{33} c_{44}) = \frac{1}{\lambda} (1-\rho) \cdot (1-\sigma) - (1-\tau)^2 \tag{14}$$

defines a set of parabolic hyperboloids in the Cartesian $\rho - \sigma - \tau$-space. Lines of equal X^2 in the planes $\tau = \tau_1$ are hyperbolae (where τ_1 is an arbitrary but fixed value of τ) with asymptotes $\rho = 1$ and $\sigma = 1$. In the quadrant $\rho < 1$, $\sigma < 1$ the set-parameter X^2 increases with increasing distance from the asymptotes.
The region of *permitted* triplets σ, ρ, τ is the open set bounded by

$$\frac{1}{\lambda} (\theta_l - \rho)(\theta_l - \sigma) = (\theta_l - \tau)^2 \tag{15}$$

$$\frac{1}{\lambda} (\theta_h - \rho)(\theta_h - \sigma) = (\theta_h - \tau)^2$$

$$\tau = \theta_l, \text{ and } \tau = \theta_h$$

For a given τ_1 the region is a lozenge-shaped area bounded by two 'constraint-hyperbolae' with asymptotes $\rho = \theta_l$, $\sigma = \theta_l$ and $\rho = \theta_h$, $\sigma = \theta_h$, respectively. The lowest and highest permitted values of X^2 correspond to those hyperbolae of the set (14) whose apex coincides with the apex of the constraint hyperbolae (15), i.e., for

$$\begin{aligned} \rho_{a,l} &= \sigma_{a,l} = \theta_l - \sqrt{\lambda}(\theta_l - \tau_1) \quad \text{and} \\ \rho_{a,h} &= \sigma_{a,h} = \theta_h - \sqrt{\lambda}(\theta_h - \tau_1) \quad \text{respectively.} \end{aligned} \tag{16}$$

Substitution of $\rho_{a,h}$ and $\sigma_{a,h}$ from (16,2) into (14) gives the lowest permitted value of X^2 for $\tau = \tau_1$, and similarly, we get for $\rho_{a,l}$ the highest permitted value of X^2 for $\tau = \tau_1$. The absolutely lowest value of X^2 is obtained for the highest permitted τ (and similarly for the absolutely highest permitted value of X^2). Thus we obtain bounds for X^2 by combining (16,2), (15,3), and (14) (or (16,1), (15,4), and (14)) for given θ_l, θ_h, and λ:

$$\frac{1-\theta_h}{\lambda}(1 - \theta_h + 2\sqrt{\lambda}(\theta_h - \theta_l) - \lambda(1 + \theta_h - 2\theta_l)) < X^2 < \frac{1-\theta_l}{\lambda}(1 - \theta_l - 2\sqrt{\lambda}(\theta_h - \theta_l) - \\ - (1 - 2\theta_h + \theta_l) \tag{17}$$

The result for constant θ is obtained immediately by setting $\theta_l = \theta_h = \theta$:

$$X_K^2 = (1-\theta)^2 \cdot \left(\frac{1}{\lambda} - 1\right) \tag{18}$$

The term $(1/\lambda - 1)$ in (18) is the relative difference of the squares of the axes of the ellipsoidal wave front of the horizontally polarized shear waves:

$$\lambda = \frac{b^2}{a^2} = \frac{c_{44}}{c_{66}}, \qquad \frac{1}{\lambda} - 1 = \frac{a^2 - b^2}{b^2} = \frac{c_{66} - c_{44}}{c_{44}} = e^2, \quad \lambda = \frac{1}{1+e^2}.$$

We re-formulate (17) in terms of e^2 and obtain

$$\begin{aligned} e^2(1-\theta_h)\left[1-\theta_l - 2(\theta_h - \theta_l)\left(\frac{1+\frac{e^2}{2} - \sqrt{1+e^2}}{e^2}\right)\right] < \\ < X^2 < e^2(1-\theta_l)\left[1-\theta_h + 2(\theta_h - \theta_l)\left(\frac{1+\frac{e^2}{2} - \sqrt{1+e^2}}{e^2}\right)\right]. \end{aligned} \tag{19}$$

The fraction in the two bounds in (19) is the series

$$\frac{1}{8}e^2 - \frac{1}{16}e^4 + \frac{5}{128}e^6 - \ldots +$$

The significant result is that for not-too-large e^2 the possible range of X^2 is much smaller than its magnitude:

$$\overline{X^2} = \frac{X_{max}^2 + X_{min}^2}{2} \approx e^2\,(1-\theta_h)\,(1-\theta_l) + \frac{1}{8}\,e^4(\theta_h - \theta_l)^2 \tag{20}$$

and

$$\Delta X^2 = \frac{X_{max}^2 - X_{min}^2}{2} \approx \frac{1}{8}\,e^4\,(\theta_h - \theta_l)\,(1-\theta_h) + (1-\theta_l)). \tag{21}$$

(20) and (21) indicate that the magnitude of the error one makes in assuming an elliptical wavefront (corresponding to $X^2 = 0$) is significantly larger than the error one can make by neglecting anisotropy (it would be zero for $X^2 = X_K^2$ according to (18)).

However, the curvature of the quasi-longitudinal wave front can be determined in a much more quantitative manner as shown in the following section.

2.2 BOUNDS FOR THE CURVATURE OF THE WAVE FRONT

We have from (2), (3), and (14)

$$\frac{r_P}{r_K} = \frac{c_{11}}{c_{33}} - \frac{4c_{44}}{(c_{33} - c_{44})} \cdot X^2 \tag{22}$$

From the conversion rules (5) we obtain

$$\frac{c_{11}}{c_{33}}=(1-2\tau)^2+\frac{4}{\lambda}(1-\sigma)\rho \quad \text{and} \tag{23}$$

$$\frac{c_{44}}{c_{33}-c_{44}}=\frac{\rho}{1-\rho}, \tag{24}$$

and thus after some algebraic manipulation

$$\frac{r_P}{r_K}=1+4(\rho-\tau)\frac{1-\tau}{1-\tau-(\rho-\tau)}=1+4h\frac{1-\tau}{1-\tau-h}, \text{ where } h=\rho-\tau. \tag{25}$$

From this general expression for the radius of curvature of the quasi-longitudinal wave front follows the 'principal theorem' for the special case $\rho=\tau$, that is, for instance, for constant θ. However, we can have $\rho=\tau$ and thus $r_P=r_K$ even for non-constant θ: this follows from the fact that (25) does not contain σ, thus σ can assume arbitrary (permitted) values, while constant θ implies $\sigma=\tau$.

Bounds for the curvature can be derived via (25) from the bounds for h. The extreme values of h follow from (15,1) and (15,2) to

$$h_{m\,1,2}=-(1-\lambda)\,\Delta\tau\pm\sqrt{(1-\lambda)\,(\Delta\theta^2-\lambda\Delta\tau^2)}, \tag{26}$$

$$\text{where } \Delta\tau=\tau-\frac{\theta_l+\theta_h}{2} \quad \text{and} \quad \Delta\theta=\frac{\theta_h-\theta_l}{2}.$$

The largest and the smallest value of h are obtained for $\Delta\tau=-\Delta\theta$ $(\tau=\theta_l)$ and $\Delta\tau=\Delta\theta\,(\tau=\theta_h)$; substitution of these values into (26) gives

$$h_{max}(\theta_l/\theta_h,\lambda)=(1-\lambda)\,(\theta_h-\theta_l),\ \tau=\theta_l \quad \text{and} \tag{27}$$

$$h_{min}(\theta_l/\theta_h,\lambda)=-(1-\lambda)\,(\theta_h-\theta_l),\ \tau=\theta_h \tag{28}$$

Inspection of the geometric relationship (see figure 1) between the surfaces of equal r_P/r_K defined by (25) (hyperbolic cylinders with the σ-axis as generator) and the sets of extremal pairs h, τ defined by (26) (these are line segments in the h-τ-plane from $(0,\theta_h)$ to (h_{max},θ_l) and from (h_{min},θ_h) to $(0,\theta_l)$ with h_{min} and h_{max} according to (27) and (28), respectively) shows that the largest radius of curvature is obtained directly by substituting (27) into (25):

$$\left.\frac{r_P}{r_K}\right|_{max}(\theta_l|\theta_h,\lambda)=1+4(\theta_h-\theta_l)\,(1-\lambda)\frac{1-\theta_l}{1-(1-\lambda)\,\theta_h-\lambda\theta_l}. \tag{29}$$

However, the corresponding relationship for the smallest radius of curvature, namely

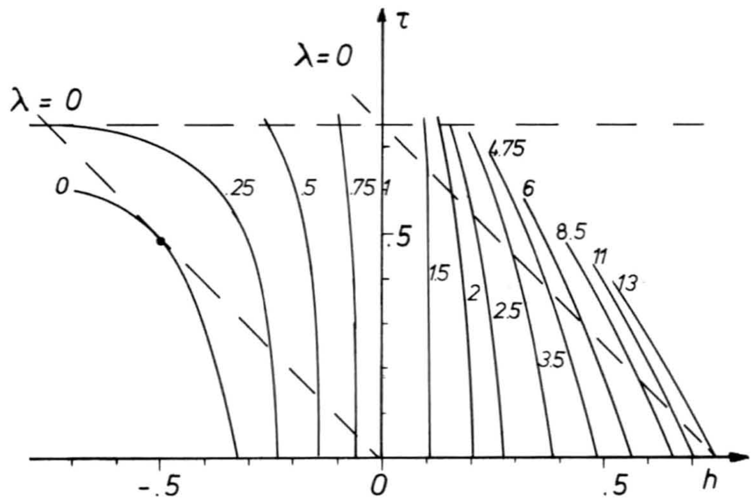

Figure 1. Lines of equal (normalized) radius of curvature in the h-τ-plane. 'Permitted' are pairs h, τ inside the parallelogram 0, $\theta_h - h_{max}$, $\theta_1 - 0$, $\theta_1 - h_{min}$, θ_h. The parallelogram for $\theta_1 = 0$, $\theta_h = \frac{3}{4}$ (i.e., for $\lambda = 0$) is indicated by broken lines. The radius of curvature is independent of λ and k, thus surfaces of equal normalized curvature are hyperbolic cylinders.

$$\frac{r_P}{r_K}\bigg|_{min}(\theta_l | \theta_K, \lambda) = 1 - 4(\theta_h - \theta_l)(1-\lambda)\frac{1-\theta_h}{1-\lambda\theta_h - (1-\lambda)\theta_l} \tag{30}$$

is valid only if at the point of intersection the slope $-d\tau/dh$ of the lines of equal $(r_P/r_K - 1)$ is at most equal to that of the line segment, i.e., if $h^2/(1-\tau)^2 < 1-\lambda$. Together with (23) this leads to the condition

$$\sqrt{(1-\lambda)}\,(\theta_h - \theta_l) \leqq (1-\theta_h). \tag{31}$$

Inequality (31) is satisfied in all practical cases, since few known media have $\theta = 0.5$ or larger, thus the left hand side is nearly always smaller than the right hand side, even for $\lambda \to 0$. However, for *theoretical* considerations (30) should not be used without checking (31). For instance, if one looks for the absolutely lowest permissible radius of curvature of the P-wave front near the axis of symmetry with the help of (30), one would get

$$\frac{r_P}{r_K}\bigg|_{min}\left(0 \,\bigg|\, \frac{3}{4}, 0\right) = 1 - 4\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{1}{4}, \text{ obtained for } \tau = \frac{3}{4},\ h = \frac{3}{4}$$

However, (31) is not satisfied, since $\frac{3}{4}$ is larger than $\left(1-\frac{3}{4}\right)$. The radius of curvature obtained in this manner is indeed not the smallest possible. The smallest turns out to be

$$\left.\frac{r_P}{r_K}\right|_{min}\left(0\left|\frac{1}{2}, 0\right.\right)=1-4\cdot\frac{1}{2}\cdot\frac{1}{2}=0, \text{ obtained for } \tau=\frac{1}{2}, h=-\frac{1}{2}.$$

This value can be accepted, since (31) is satisfied. The highest possible value of $\frac{r_P}{r_K}$ is obtained for $\tau=0$, $h=.75$:

$$\left.\frac{r_P}{r_K}\right|_{max}\left(0\left|\frac{3}{4}, 0\right.\right)=1+4\cdot\frac{3}{4}\frac{1}{1-\frac{3}{4}}=13.$$

The corresponding ellipsoid has infinite radius of curvature (because $\lambda\to 0$).

Equations (29) and (30) constitute bounds on the radius of curvature in stratified media with constituents with θ in the (closed) interval $[\theta_l, \theta_h]$. Such bounds are still unsatisfactory if λ is not known. If the highest and lowest μ are μ_l and μ_h, respectively with $\mu_h>\mu_l$, we have as bounds for λ

$$1-\left(\frac{\mu_h-\mu_l}{\mu_h+\mu_l}\right)^2\leqq\lambda<1. \tag{32}$$

The equality sign prevails if μ_l and μ_h contribute exactly $\frac{1}{2}$ of the compound medium. Thus we finally have

$$\begin{aligned}-4(\theta_h-\theta_l)\left(\frac{\mu_h-\mu_l}{\mu_h+\mu_l}\right)^2\frac{1-\theta_h}{1-\left(\frac{\Delta\mu}{\mu_m}\right)^2\theta_l-\left(1-\left(\frac{\Delta\mu}{\mu_m}\right)^2\right)\theta_h}<\frac{r_P-r_K}{r_K}<\\<4(\theta_h-\theta_l)\left(\frac{\Delta\mu}{\mu_m}\right)^2\frac{1-\theta_l}{1-\left(\frac{\Delta\mu}{\mu_m}\right)^2\theta_h-\left(1-\left(\frac{\Delta\mu}{\mu_m}\right)^2\right)\theta_l}\end{aligned} \tag{33}$$

For example, with $\mu_h/\mu_l=4$, $\theta_l=0{,}25$, $\theta_h=0.35$, we have

$$0.864<\frac{r_P}{r_K}<1.151.$$

For the radius of curvature of ellipsoids with the same axes (and overestimated $1-\lambda$) we would have

$$1.29<\frac{r_E}{r_K}<1.67.$$

Inequality (33) – or its equivalent with $[(\mu_h - \mu_l)/(\mu_h + \mu_l)]^2$ replaced by $1 - \lambda$ – does not contain the curvature of the ellipsoid with the same axes at all. Though at the beginning of this chapter we started with the fundamental inequality – which can be read as an inequality containing r_P/r_K and r_E/r_K – the second term was lost on the way (in the step from (23) to (25)). What we have obtained is a reformulation of the 'principal theorem' allowing for a range of θ instead for constant θ.

3. THE SLOWNESS SURFACE OF THE K-MEDIUM AND ITS RELATION TO THE VELOCITY DETERMINED FROM REFLECTION TIME MEASUREMENTS

3.1 THE SLOWNESS SURFACE OR THE K-MEDIUM

The ray geometric properties of an anisotropic medium can be represented by anyone of its characteristic surfaces. These are representations in polar coordinates of the velocities and their inverse (the slownesses) of plane waves and rays, respectively. For transversely isotropic media a section of these surface containing the axis of symmetry is sufficient. The four surfaces can be converted into each other by the operations 'inversion' (mapping by inverse radii), 'tangent surface/first pedal surface', and 'polar reciprocity', if we ignore simple changes of scale (fig. 2). Polar reciprocity exists between surfaces F_1 und F_2 if to each point A_1 on F_1 there exists a point A_2 on F_2 such that the radius vector from the origin to A_1 is parallel to the surface normal in A_2, and vice versa.

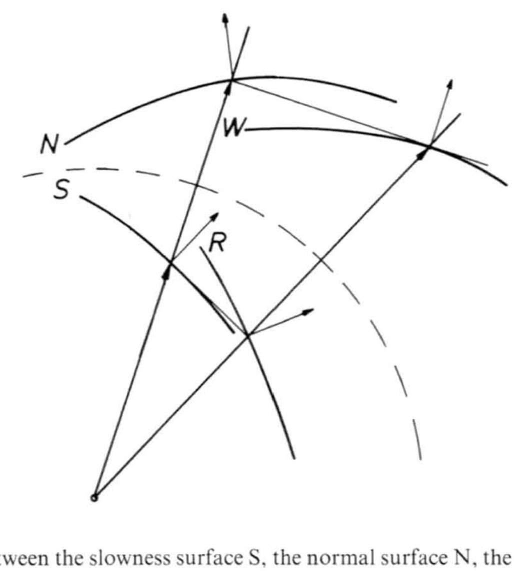

Figure 2. Relation between the slowness surface S, the normal surface N, the wave surface W, and the ray slowness surface R. The pairs N, S and W, R are inverses of each other (i.e., "reciprocal radii" map N into S, S into N, W into R, and R into W); N is the first pedal of W (i.e., the locus of plumb lines from the origin onto the tangent planes of W), and R is the first pedal of S. Since both mapping by reciprocal radii and formation of pedals preserve angles, S and W are polar reciprocals of each other (in corresponding points the normal to S is parallel to the radius vector of W and vice versa).

Notwithstanding the complete equivalence of the four surfaces, the slowness surface is to be preferred by far: not only is the mathematical representation simpler than for the other surfaces, it also embodies the most important parameters – for instance those needed for the determination of refracted and reflected rays at interfaces – in the most convenient form. In Krey and Helbig (1956) a parameter representation for the slowness surface of quasi-longitudinal wave in K-media was given that contained only two parameters related to the medium – Q, the square of the ratio of the velocity in horizontal and vertical direction, and another parameter that can be expressed through Q and Poisson's ratio. Since a K-medium has only two dimensionless elastic contants, it should be possible to express *all* ray geometric properties in terms of two arbitrarily chosen parameters, for instance in terms of

$$\tau = \rho = (V_{S,\perp}/V_{P,\perp})^2$$

and

$$\lambda = (V_{SH,\perp}/V_{SH,\parallel})^2,$$

where the symbols $\perp$ and $\parallel$ designate propagation at right angles to and parallel to the plane of stratification, respectively. In this form the normalized slowness surface is described by

$$n^2_{P,K} = \frac{1+(2+4\tau e^2)w+w^2}{1+(2+4\tau e^2)w+(1+4\tau e^2(1-\tau))w^2} \tag{34}$$

and

$$tg^2\beta = \frac{w+w^2}{1+(1+4\tau e^2)w}, \quad \text{where again } e^2 = \frac{1}{\lambda}-1. \tag{35}$$

The curve parameter w is related to the direction of the wave normal and the direction of particle displacement by

$$w = tg(\alpha) \cdot tg(\beta), \tag{36}$$

where α is the angle between the (quasi-longitudinal) displacement direction and the axis of symmetry and β is the angle between the wave normal and the axis of symmetry.

To obtain the slowness in direction β, w is determined from (35) and substituted into (34).

The other characteristic surfaces can be obtained in the following way:

a) normal velocity surface: $v(w) = 1/(n(w))$.

b) the ray slowness surface is the first pedal surface of the slowness surface, thus

$$n_{ray}(w) = n(w) \cdot \cos(\gamma(w) - \beta(w)), \tag{37}$$

where γ is the angle between the normal to the slowness surface (i.e., the direction of the ray) and the axis of symmetry.

c) The wave surface is the locus of the endpoints of the ray-velocity (or group velocity) vectors. It is the inverse of the ray-slowness surface, thus

$$g(w) = \frac{1}{n(w) \cdot \cos(\gamma(w) - \beta(w))}. \tag{38}$$

For (37) and (38) we need γ, which can be obtained from

$$\operatorname{tg}\gamma = -\frac{dn_3/dw}{dn_1/dw} = -\operatorname{tg}\beta \frac{d(n_3^2)/dw}{d(n_1^2)/dw}, \tag{39}$$

where n_1 and n_3 are $n \cdot \sin\beta$ and $n \cdot \cos\beta$, respectively. For the K-medium we obtain

$$\operatorname{tg}\gamma/\operatorname{tg}\beta = 1 + 8\tau(1-\tau) \cdot e^2 \cdot w \frac{1 + 2\tau \cdot e^2 \cdot w}{1 + 2w + (2 + 4\tau e^2 - \tau)w^2}. \tag{40}$$

3.2 THE t^2 vs x^2 PLOT FOR A HOMOGENEOUS K-MEDIUM

The signal from a virtual source at $x = 0$, $z = 1$ arrives at detector position $(x, 0)$ at time $t = \sqrt{(1 + x^2)}/g(\gamma)$, where $\gamma = \tan^{-1}(x)$. Thus we have as parameter representation of the t^2 vs x^2 plot

$$x^2 = \frac{w + w^2}{1 + (1 + 4\tau e^2)w} \cdot \left[1 + 8\tau(1-\tau) \cdot e^2 \cdot w \frac{1 + 2\tau e^2 w}{1 + 2w + (2 + 4\tau e^2 - \tau)w^2}\right]^2 \tag{41}$$

and

$$t^2 = (1 + x^2) \cdot \frac{1 + (2 + 4\tau e^2)w + w^2}{1 + (2 + 4\tau e^2)w + (1 + 4\tau e^2(1-\tau)w^2} \cos^2(\gamma - \beta) \tag{42}$$

with

$$\cos^2(\gamma - \beta) = \frac{1}{1 + \operatorname{tg}(\gamma - \beta)} \tag{43}$$

and

$$\operatorname{tg}(\gamma - \beta) = \frac{\operatorname{tg}\gamma - \operatorname{tg}\beta}{1 + \operatorname{tg}\gamma \operatorname{tg}\beta} = \frac{\operatorname{tg}\beta \cdot \left(\dfrac{\operatorname{tg}\gamma}{\operatorname{tg}\beta} - 1\right)}{1 + \operatorname{tg}^2\beta \dfrac{\operatorname{tg}\gamma}{\operatorname{tg}\beta}}. \tag{44}$$

Substitution of (35) and (40) into (44) and of the result into (43) and then into (42) leads to an analytical parameter-expression for the t^2 vs x^2 plot over a K-medium. Unfortunately, the resulting expressions are very unwieldy. However, a numerical evaluation is not difficult. The curve labeled $v_P^2(x^2)$ in figure 3 was obtained in this way for $\tau = 0.3$ and $e^2 = 1.33\ldots$.

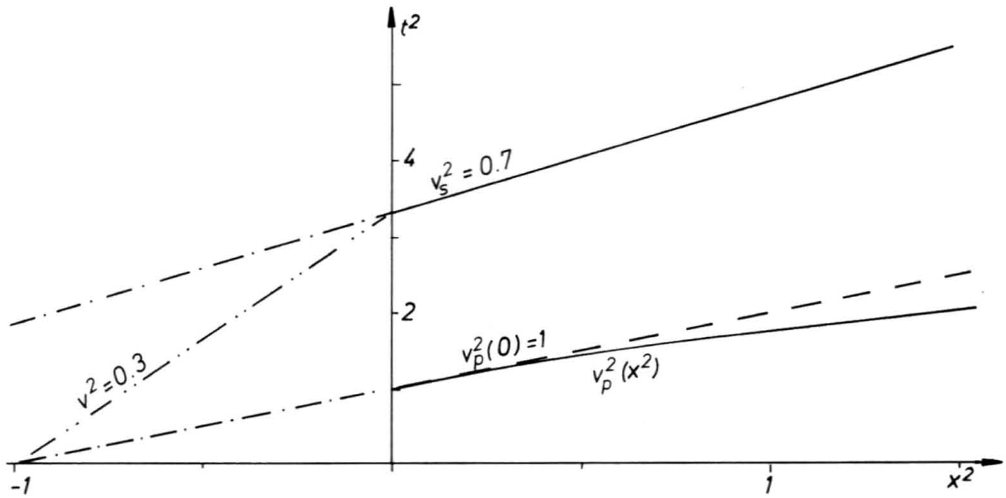

Figure 3. Hypothetical t^2 – vs – x^2 plot for P and SH wave from virtual source at $z = 1$ for $\lambda = 3/7$ and $\tau = 0.3$.

3.3 AN APPLICATION TO THE COMBINED USE OF LONGITUDINAL AND TRANSVERSE WAVES

Figure 3 shows a t^2 vs x^2 plot of two reflections, one from an SH-experiment and one from a P-experiment. The fact that the SH-branch is straight while the P-branch is convex upwards indicates that the medium is homogeneous and transversely isotropic (in an inhomogeneous medium both branches would be curved, in a homogeneous isotropic medium both would be straight). Standard velocity interpretation would lead to the assumption that the two branches could not be caused by the same reflector, since then $t^2(o) \cdot v^2 = z^2$ should give the same result in both cases (i.e., the backward-extrapolation of the tangents to the two curves at $x^2 = 0$ should intersect the x^2-axis at $-z^2$). The reason for this discrepancy is simple: the SH-front is elliptical, and thus the t^2 vs x^2 line gives the velocity in *horizontal* direction, while the relationship $z^2 = t_o^2 \cdot x^2/(t^2 - t_o^2) = t_o^2 \cdot v_{stack}^2$ is valid only if v_{stack} is a measure for the velocity in vertical direction.

If more is known about the medium, one can make use of the internal relationship between the parameters of the different sheets of the wave surface to obtain information on the ratio of the axes of the SH-ellipsoids. How this relationship could be utilized is shown for a K-medium. The ratio of the slopes of the regression lines one observes is

$$\psi = c_{66}/c_{33} = \tau/\lambda,$$

the ratio we need is $c_{44}/c_{33} = \tau$.

We can now plot any observable quantity affected by τ and λ for the observed ψ and a set of values of τ and compare the 'master curves' thus obtained with the observations. It is obvious that the effects can, at best, be small, therefore several such matchings might be necessary. For the present numerical example (with arbitrarily high accuracy) $\tilde{V}_P(X)$, the square root of the slope of the tangent to the t^2 vs x^2-curve was used. The slope was obtained from (41)–(44) with small stepsize ($\Delta x^2 = .0355$ at $x = 1$), $\psi = .7$ and $\tau = .2$, .225, .25, .275, .3, .325, .35, .375, and .4. Fig. 4 shows the quantity $(v_P(x) - v_P(0))/v_P(0)$ as function of x. As to be expected in a numerical example of this type, the curve for $\tau = .3$ ($v_{S\perp}/v_{P\perp} = .548$, $\lambda = .429$) matches. What is important is not this agreement, but the estimate that, in order to obtain the velocity ratio with an accuracy of 5 percent, the slope must be reliable within about 6 percent at an offset equal to twice the depth of the reflector.

It goes without saying that this is not meant as a suggestion for practical application. One of the complications is that we rarely are faced with homogeneous medium. It might be possible to separate the inhomogeneity-induced curvature from anisotropy-

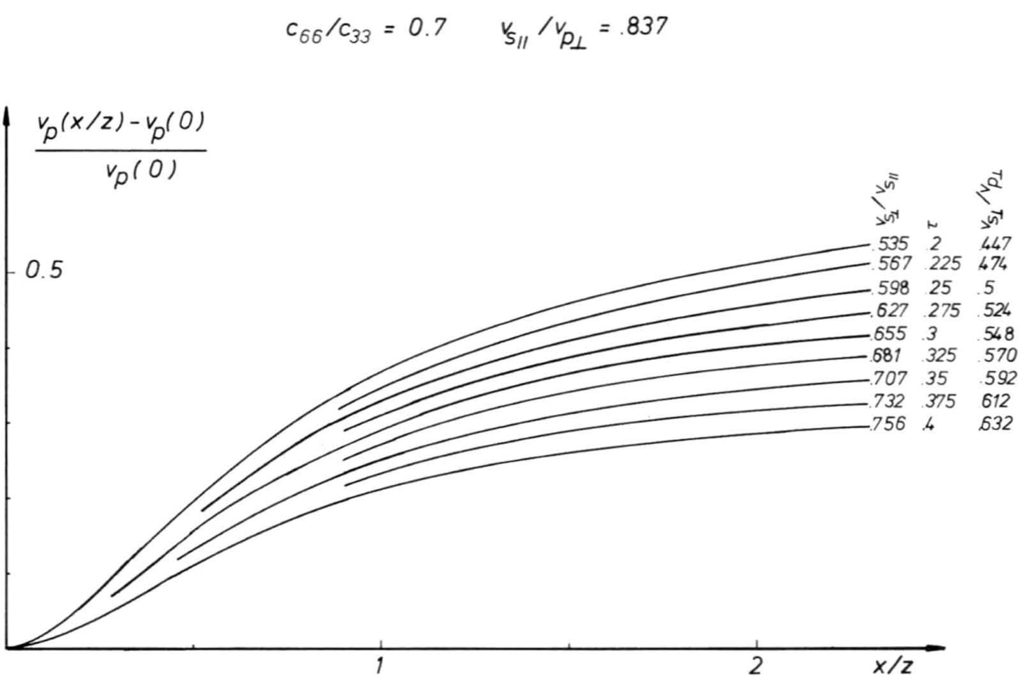

Figure 4. Set of curves representing square root of inverse slope of the $t^2 - \mathrm{vs} - x^2$ curve for $\lambda \cdot \tau = c_{66}/c_{33} = 0.7$ and $.2 < \tau \leqslant .4$. The curve of figure 3 corresponds to the curve labled $\tau = .3$.

induced curvature with the help of the SH-branch, but this would reduce the accuracy even further. I also do not intend to create the impression that the slope of the t^2 vs x^2 curve – or its square root – is the most suitable parameter for such calculations. It is to be hoped that there are other parameters that can be observed more directly and applied more conveniently.

We might question how well the assumption of a K-medium is satisfied in a particular situation. If a range of θ has to be taken into account, the tight relation between the parameters of the different sheets of the slowness surface (and the other characteristic surfaces) is relaxed, and the above strict relations have to be replaced by bounds (similar to the derivation in section 2.1), and only if the intersection of all bounds used becomes sufficiently small can the general method be applied.

4. CONCLUSIONS

The original theorem was valid for K-media only, i.e., for layer sequences where all constituents had the same ratio of transverse to longitudinal velocity. No estimate existed for the degree of deviation from a sphere for situations where this condition was not (or not exactly) satisfied. The bounds supplied by the original inequality, on the other hand, were so wide that their only significance was to relegate the elliptical wave front for P-waves to the status of an unattainable limit. With the re-formulation presented here, the bounds are those compatible with the actually occuring range of velocity ratios (instead of the range permitted by stability considerations), and the theorem is now nothing but the special case of the inequality: if the range degenerates to a single value, the upper and the lower bound coincide.

The new formulation of the bounds not only allows to estimate the deviation of the P-wave front from a sphere for layer sequences with a range of velocity ratios for the constituents, but also allow to establish limiting relationships between the geometric shapes of the P-wave front and the SH-wave front (for K-media this is a strict relationship). Such relationship might be used in combining P-wave- and SH-wave observations, but it remains to be seen whether this approach leads to practical results.

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